ROTA'S UNIVERSAL OPERATORS AND INVARIANT SUBSPACES IN HILBERT SPACES

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Abstract. A Hilbert space operator is called *universal* (in the sense of Rota) if every operator on the Hilbert space is similar to a multiple of the restriction of the universal operator to one of its invariant subspaces. We exhibit an analytic Toeplitz operator whose adjoint is universal in the sense of Rota and commutes with a quasi-nilpotent injective compact operator with dense range. In particular, this new universal operator invites an approach to the *Invariant Subspace Problem* that uses properties of operators that commute with the universal operator.

1. Introduction

Questions concerning the existence of invariant subspaces for particular classes of operators have produced a wealth of interesting theorems and examples, successfully linking branches of analysis such as Harmonic Analysis, Complex Function Theory, and Functional Analysis. In this respect, the first known result traces back more than half a century ago, when in the fifties von Neumann proved, in an unpublished manuscript, that every non-zero linear compact operator, K, acting on a Hilbert space $\mathcal{H}$ has a nontrivial closed invariant subspace M, that is, $M \neq \{0\}$, $M \neq \mathcal{H}$ and for v in M, its image, Kv, is also in M.

Later, in 1954, Aronszajn and Smith [2] extended von Neumann's Theorem to the Banach space setting. By the end of the sixties, Bernstein and Robinson [3] and Halmos [19] proved the analogous result for polynomially compact Hilbert space operators.

In 1973, Lomonosov [21] proved a remarkable theorem that probably, up to this point, is the main affirmative result for operators on general Banach spaces: *any linear bounded operator T, not a multiple of the identity, has a nontrivial invariant closed subspace if it commutes with a non-scalar operator that commutes with a nonzero compact operator*. A couple of years later, though its publication was delayed for more than ten years, Enflo showed the existence of a separable Banach space and a linear bounded operator T without nontrivial closed invariant subspaces. Enflo's construction was ingenious and very difficult; the main idea was

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to start with the operator of multiplication by the independent variable on the space of polynomials $\mathcal{P}$ and construct a norm on the space so that every non-zero vector is cyclic for the extension of the operator to the completion of the polynomials (see [13]).

In 1985, Read [26] constructed a bounded linear operator without nontrivial closed invariant subspaces in the well-known sequence space ℓ^1. His construction, which appears to be the first example of such an operator on any of the classical Banach spaces, was simpler than Enflo's in some sense. Read, subsequently, made an even more remarkable construction of a bounded linear operator on ℓ^1 that has no closed invariant *sets* except the trivial ones (see [27]).

More recently, a rather striking affirmative result in the Banach space context has been provided by Argyros and Hydon [1], who constructed an infinite dimensional Banach space such that every linear bounded operator is a compact perturbation of a scalar operator. Therefore, by Lomonosov's Theorem, every linear bounded operator in this Banach space has a nontrivial closed hyperinvariant subspace.

In the Hilbert space setting, in 1978 Brown had established the existence of nontrivial invariant subspaces for subnormal operators [4]. His seminal ideas, based on Sarason's study of the weak-star density of polynomials to characterize those normal operators that are reduced by all of their invariant subspaces [31], led to a variety of further applications (see also [5]).

The *Invariant Subspace Problem* may, therefore, be considered one of the most prominent open problems in the study of linear bounded operators on separable Hilbert spaces; and there have been many significant developments in this branch of Operator Theory (see the classical book by Radjavi and Rosenthal [25] and the recent one by Chalendar and Partington [7]).

A remarkable approach to the problem is related to the transformations Rota [28] called universal operators: an operator U on a Hilbert space $\mathcal{H}$ is called *universal* if for any linear bounded A on $\mathcal{H}$, there exist a non-zero complex constant λ and a closed invariant subspace M of U such that the restriction of U to M is similar to λA. In other words, understanding the *Invariant Subspace Problem* on Hilbert spaces becomes a question of understanding the invariant subspaces of the single operator U. Note that, in particular, every bounded linear operator on an infinite dimensional separable Hilbert space $\mathcal{H}$ would have a nontrivial closed invariant subspace if and only if any minimal invariant subspaces of a universal operator U on $\mathcal{H}$ are just one-dimensional.

The best known example of a universal operator is the adjoint of a unilateral shift of infinite multiplicity. In the eighties, Nordgren, Rosenthal and Wintrobe [23] proved that if φ is a hyperbolic automorphism of the unit disc and μ is in the interior of the spectrum of the composition operator C_φ acting on the classical Hardy space H^2, then $C_\varphi - \mu I$ is a universal operator on H^2. Of course, the lattices of the closed invariant subspaces of $C_\varphi - \mu I$ and C_φ coincide so they have the same the minimal invariant subspaces. In [17], Gallardo-Gutiérrez and Gorkin studied the behavior of the Hardy functions in order to determine when the cyclic subspaces under $C_\varphi - \mu I$ generated by them are minimal. In the authors' paper [11], it was shown that C_φ is similar to the adjoint of an analytic Toeplitz operator T_ψ^* whose symbol ψ is a covering map of an annulus, behaving, in some sense, like the adjoint of the shift of infinite multiplicity.

Nevertheless, a different point of view in the context of Rota's universal operators will be taken in the present work. Our main aim is to link Lomonosov Theorem to the context of Rota's universal operators, asking whether there exists a universal operator that commutes with an *interesting compact operator*, i.e., an injective compact operator with dense range (note that, using direct sums of operators, trivial examples may be provided to get universal operators commuting with compacts).

In addition, let us remark that, to our knowledge, none of the *standard* universal operators (in the sense of Rota) known in the literature can commute with an injective compact operator with dense range. Indeed, in the first author's papers [8, Thm. 10] and [9], it is shown that neither the Nordgren, Rosenthal and Wintrobe operator nor the adjoint of a unilateral shift of infinite multiplicity can commute with a nontrivial compact operator.

Moreover, if S is an analytic Toeplitz operator on the Hardy space H^2 whose symbol is a singular inner function or infinite Blaschke product, it is straightforward that S is an isometric operator and S^* has infinite dimensional kernel mapping H^2 onto H^2. Caradus' Theorem [6] yields that S^* is a universal operator. Using the Wold decomposition, such an operator can be represented as a block matrix on $\mathcal{H} = \oplus_{k=0}^{\infty} S^k \mathcal{W}$ where $\mathcal{W} = H^2 \ominus SH^2$ that is upper triangular and has the identity on the super-diagonal:

$$S^* \sim \begin{pmatrix} 0 & I & 0 & 0 & \cdots \\ 0 & 0 & I & 0 & \cdots \\ 0 & 0 & 0 & I & \cdots \\ & & & \ddots & \end{pmatrix}$$

An easy computation shows that every operator that commutes with S^* has the form

$$A \sim \begin{pmatrix} A_0 & A_{-1} & A_{-2} & A_{-3} & \cdots \\ 0 & A_0 & A_{-1} & A_{-2} & \cdots \\ 0 & 0 & A_0 & A_{-1} & \cdots \\ & & & \ddots & \end{pmatrix}$$

an upper triangular block Toeplitz matrix, that is, an upper triangular block matrix whose entries on each diagonal are the same operator on the infinite dimensional Hilbert space $\mathcal{W}$. Because every block in such a matrix occurs infinitely often, it is easy to see that the only compact operator that commutes with the universal operator S^* is 0, not an interesting compact operator.

Because Lomonosov's Theorem connects commuting with a nontrivial compact operator with the existence of invariant subspaces, it seems reasonable that a universal operator that commutes with a nontrivial compact operator might be helpful in proving the existence of invariant subspaces.

In this sense, our main result in this work will be

Main Theorem. *There exists a universal operator for separable, infinite dimensional Hilbert spaces that commutes with an injective compact operator with dense range.*

As a consequence, we will describe proper invariant linear manifolds (not necessarily closed) for any linear bounded operator acting on a separable, infinite dimensional Hilbert space.

Concerning the structure of this paper, we first state the Caradus Theorem [6] that gives a sufficient condition for an operator to be universal in the sense of Rota [28]. Then, on the separable, infinite dimensional Hilbert space H^2, we construct an analytic Toeplitz operator and use the Caradus Theorem to prove that the adjoint of the analytic Toeplitz operator is universal. In Section 3, we construct a weighted composition operator and we outline the proof that its adjoint is an injective compact operator with dense range that commutes with the universal operator just constructed. In Section 4, some of the important properties of these operators that we need are described, including the properties used in the proof of the properties of the weighted composition operator needed in Section 3. Finally, in Section 5, we apply these operators to produce closed invariant subspaces for operators in a few cases and invariant linear manifolds in all cases.

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2. A New Example of a Universal Operator

The major work in this paper is set in the Hardy Hilbert space, $H^2(\mathbb{D})$ (also written H^2). Of course, because any two separable, infinite dimensional complex Hilbert spaces are isometrically isomorphic, our choice of H^2 is not limiting in any way. There are two standard definitions for $H^2(\mathbb{D})$; the power series definition is

$$H^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ where } \|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

If we regard the series for f as a Fourier series $\sum_{n=0}^{\infty} a_n e^{in\theta}$, then we see how $H^2(\mathbb{D})$ can be regarded as the closed subspace of $L^2(\partial\mathbb{D})$ consisting of those functions whose negative Fourier coefficients are all 0.

The second definition connects $H^2(\mathbb{D})$ with $L^2(\partial\mathbb{D})$ via integration:

$$H^2(\mathbb{D}) = \{f \text{ analytic in } \mathbb{D} : \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty\}$$

From this perspective, $\|f\|^2$ is the supremum in the above definition and the norm for H^2 is the same using either definition. $H^2(\mathbb{D})$ is a "Hilbert space of analytic functions" in the sense of [12]: in particular this means that for f in H^2, the map $f \mapsto f(\alpha)$ is a continuous linear functional for each α in the unit disk. It is well known that the kernel functions on H^2 are $K_\alpha(z) = (1-\overline{\alpha}z)^{-1}$ for α in $\mathbb{D}$. This means for any f in H^2, $\langle f, K_\alpha\rangle = f(\alpha)$ where $\langle\cdot,\cdot\rangle$ is the inner product on H^2.

The following theorem gives a prescription for finding universal operators on Hilbert spaces.

Theorem 1. (Caradus [6, p. 527] or see [7, p. 214]) *If $\mathcal{H}$ is a separable Hilbert space and U is a bounded operator on $\mathcal{H}$ such that:*

(1) *The null space of U is infinite dimensional.*
(2) *The range of U is $\mathcal{H}$.*

Then U is universal for $\mathcal{H}$.

We begin by describing the construction of the operators we want to use, an analytic Toeplitz operator and a weighted composition operator.

For ϕ a bounded analytic function on the unit disk, that is, ϕ is in $H^\infty(\mathbb{D})$, the analytic Toeplitz operator, T_ϕ, on H^2 is the operator defined by $(T_\phi h)(z) = \phi(z)h(z)$ for h in H^2. For ϕ a bounded analytic function on the disk, T_ϕ is a bounded operator on H^2 and it is easy to prove that $\|T_\phi\| = \|\phi\|_\infty$. More generally, if f is a function in $L^\infty(\partial\mathbb{D})$, the Toeplitz operator T_f is the operator on H^2 given by $T_f h = P_+ fh$ where P_+ is the orthogonal projection from $L^2(\partial\mathbb{D})$ onto H^2 and h is a function in H^2. In the case that ϕ is in H^∞, the projection P_+ has no effect: for h in H^2 and ϕ in H^∞, $P_+\phi h = \phi h$. Douglas's book [14] can provide some background on properties of Toeplitz operators.

For J an analytic map of the unit disk into itself, the composition operator, C_J, on H^2 is the operator defined by $(C_J h)(z) = h(J(z))$. The boundedness of C_J for any analytic function J mapping the unit disk into itself is a consequence of the Littlewood Subordination Theorem [20] (or see [12, pp. 30 & 117]). If ψ is in $H^\infty(\mathbb{D})$ and J is an analytic map of the disk to itself, the weighted composition operator, $W_{\psi,J}$, on H^2 is the operator $W_{\psi,J} = T_\psi C_J$, also a bounded operator.

To begin the construction, let $\Omega = \{z \in \mathbb{C} : \operatorname{Im} z^2 > -1 \text{ and } \operatorname{Re} z < 0\}$, which is the region in the second quadrant of the complex plane above the branch of the hyperbola $2xy = -1$. Let σ be the Riemann map of $\mathbb{D}$ onto Ω defined by

$$\sigma(z) = \frac{-1+i}{\sqrt{z+1}} \tag{1}$$

where we choose the branch of $\sqrt{\cdot}$ on the halfplane $\{z : \operatorname{Re} z > 0\}$ satisfying $\sqrt{1} = 1$. Notice that $\sigma(1) = (-1+i)/\sqrt{2}$, $\sigma(0) = -1+i$, and $\sigma(-1) = \infty$. We define ϕ on the unit disk by

$$\phi(z) = e^{\sigma(z)} - e^{\sigma(0)} = e^{\sigma(z)} - e^{-1+i} \tag{2}$$

It will be helpful to point out some of the properties of σ, e^σ, and ϕ. We will use the set $\Gamma = \{e^{i\theta} : -\pi < \theta < \pi\}$, the unit circle except -1, in this description.

[1] $\Omega = \sigma(\mathbb{D})$ is the region in the second quadrant of the complex plane above the branch of the hyperbola $2xy = -1$ and this branch is $\sigma(\Gamma)$. Moreover, $\sigma(0)$ is not on the curve $\sigma(\Gamma)$.

Proof of [1]: The map $z \mapsto 1/(z+1)$ takes the unit disk onto the half plane $\{w : \operatorname{Re}(w) > 1/2\} \subset \{w : -\pi/2 < \arg(w) < \pi/2\}$ with the curve Γ going to the line $\operatorname{Re}(w) = 1/2$, where $\{e^{i\theta} : -\pi < \theta < 0\}$ is mapped to

$\{1/2+iy : y>0\}$ and the other half of the circle to the other half of the line. This means $z \mapsto 1/\sqrt{z+1}$ maps the disk into a subset of $\{w : -\pi/4 < \arg(w) < \pi/4\}$ and the point $x+iy$ is in the image of the disk under this map if and only if $(x+iy)^2$ satisfies $\mathrm{Re}\left((x+iy)^2\right) > \frac{1}{2}$. In other words, the image of the disk is $\{w = x+iy : x^2 - y^2 > \frac{1}{2}\}$, the region to the right of the $x>0$ branch of the hyperbola $x^2 - y^2 = \frac{1}{2}$ which has asymptotes $y = \pm x$. Now, $\sigma(\mathbb{D})$ is $(-1+i) = \sqrt{2}e^{3\pi i/4}$ times this set, so $\sigma(\mathbb{D}) \subset \{w : \pi/2 < \arg(w) < \pi\}$ and it is the set of points above the branch corresponding to $y>0$ of the hyperbola $2xy = -1$ which has the negative real and positive imaginary axes as asymptotes. Since σ is analytic and univalent in $\{z : \mathrm{Re}\, z > -1\}$, which contains $\mathbb{D}$ and Γ, the curve $\sigma(\Gamma)$ is the boundary, in $\mathbb{C}$, of $\sigma(\mathbb{D})$. Moreover, since 0 is not on Γ, the point $\sigma(0)$ is not on the curve $\sigma(\Gamma)$.

[2] The function e^σ maps the curve Γ onto a curve spiraling out from the origin and tending asymptotically to $\partial\mathbb{D}$ (see Figure 1). In particular, for each r with $0<r<1$, the circle of radius r centered at the origin intersects the curve $e^{\sigma(\Gamma)}$ in exactly one point. Moreover, the closure of $e^{\sigma(\Gamma)}$ is the set $\{0\} \cup \sigma(\Gamma) \cup \partial\mathbb{D}$ so the distance of $e^{\sigma(0)}$ from this closed set is positive.

Proof of [2]: Since the lower half of the unit circle, $\{e^{i\theta} : -\pi < \theta < 0\}$, is mapped by $1/(z+1)$ to the ray $\{1/2+iy : y>0\}$, σ maps the lower half of the circle to the part of the hyperbola $\sigma(\Gamma)$ that is asymptotic to the negative real axis and we see that $\lim_{\theta \to -\pi^+} e^{\sigma(e^{i\theta})} = 0$. As θ increases from $-\pi$ to π, both the real part and the imaginary part of $\sigma(e^{i\theta})$ increase, the real part from $-\infty$ to 0 and the imaginary part from 0 to $+\infty$. This implies that $|e^{\sigma(e^{i\theta})}|$ increases from 0 to 1 and the argument of $e^{\sigma(e^{i\theta})}$, however normalized, increases continually and covers an unbounded interval of $\mathbb{R}$. In particular, for each r with $0<r<1$, there is exactly one θ for which $|e^{\sigma(e^{i\theta})}| = r$ and this is the point where the circle of radius r centered at the origin intersects the curve $e^{\sigma(e^{i\theta})}$. It also follows that, for $0<\delta<1$, on every interval $\pi - \delta < \theta < \pi$, the curve $e^{\sigma(e^{i\theta})}$ goes around 0 infinitely often while $|e^{\sigma(e^{i\theta})}|$ increases from $|e^{\sigma(e^{i(\pi-\delta)})}|$ to 1, so $\partial\mathbb{D}$ is in the closure of $e^{\sigma(\Gamma)}$. Since $e^{\sigma(0)}$ is not on $e^{\sigma(\Gamma)}$ and $0 < |e^{\sigma(0)}| < 1$, the distance from $e^{\sigma(0)}$ to the closure of $e^{\sigma(\Gamma)}$ is positive.

[3] The function e^σ is an infinite-to-one map of the unit disk, $\mathbb{D}$, onto $\mathbb{D} \setminus \{0\}$.

Proof of [3]: The exponential function does not have 0 in its range, so e^σ does not have 0 in its range. For every r with $0<r<1$, by part [2] above, the curve $e^{\sigma(\Gamma)}$ intersects the circle of radius r with center at 0 exactly once. This means there is θ_r with $-\pi < \theta_r < \pi$ so that $|e^{\sigma(\theta_r)}| = r$. Now the ray $\{z = \sigma(\theta_r) + iy : y > 0\}$ is contained in $\Omega = \sigma(\mathbb{D})$. Since the real parts of each of the numbers on this ray are the same, each point of the ray is mapped by the exponential function onto the circle of radius r. Since z in the ray implies $z + 2n\pi i$ is on the ray for every positive integer n, the exponential function maps the ray infinite-to-one onto the circle. Since the union of these circles is exactly $\mathbb{D} \setminus \{0\}$, this proves [3].

[4] The function $\phi = e^\sigma - e^{-1+i}$, defined in Equation (2), is bounded below on Γ, and therefore is an invertible function in $L^\infty(\partial\mathbb{D})$.

Proof of [4]: Clearly $\|\phi\|_\infty \le 1 + e^{-1}$ because $e^{\sigma(\Gamma)} \subset \mathbb{D}$. By [2] above, the distance from $e^{\sigma(\Gamma)}$ to e^{-1+i} is positive. This means the function $1/\phi$ is also in $L^\infty(\partial\mathbb{D})$ and ϕ is bounded below on Γ.

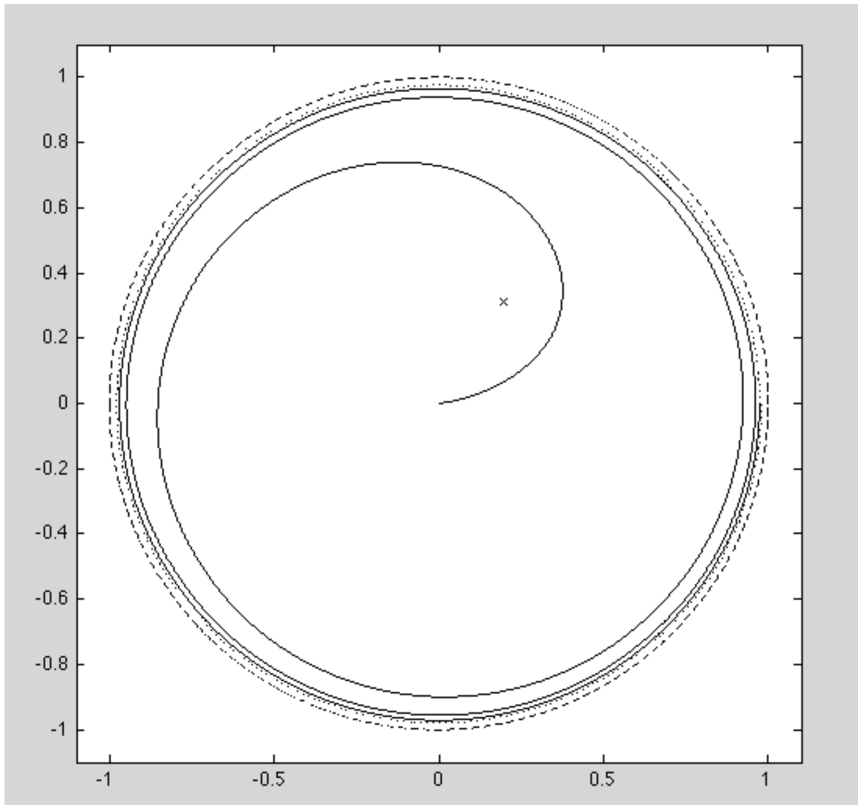

FIGURE 1. The set $e^{\sigma(\partial\mathbb{D})}$ with $\partial\mathbb{D}$ and $e^{\sigma(0)}$.

Lemma 2. *If f is a function in $H^\infty(\mathbb{D})$ and there is $\ell > 0$ so that $|f(e^{i\theta})| \ge \ell$ almost everywhere on the unit circle, then $1/f$ is in $L^\infty(\partial\mathbb{D})$ and the (non-analytic) Toeplitz operator $T_{1/f}$ is a left inverse for the analytic Toeplitz operator T_f.*

Proof. It is well known (for example, see [14]), that if g and h are in H^∞, then $T_gT_h = T_{gh}$ and $T_{\overline{g}}T_h = T_{\overline{g}h}$.

Let $f = \eta f_0$, with η inner and f_0 outer, be the inner-outer factorization of f. Since, $|f| = |f_0|$ is bounded away from 0 on $\partial\mathbb{D}$, we know $1/f_0$ is also an outer function in H^∞. On $\partial\mathbb{D}$, the $L^\infty(\partial\mathbb{D})$ function $1/f = 1/(\eta f_0) = \overline{\eta}/f_0$, so

$$T_{1/f}T_f = T_{\overline{\eta}/f_0}T_{\eta f_0} = T_{\overline{\eta}}T_{1/f_0}T_{f_0}T_\eta = T_{\overline{\eta}}T_\eta = T_{\overline{\eta}\eta} = I$$

□

Corollary 3. *The analytic Toeplitz operator T_ϕ has a left inverse.*

Corollary 4. *The Toeplitz operator T_ϕ^* has a right inverse and T_ϕ^* maps $H^2(\mathbb{D})$ onto itself.*

Proof. We have $T_{1/\phi}T_\phi = I$, so $T_\phi^* T_{1/\phi}^* = (T_{1/\phi}T_\phi)^* = I$. This equality implies T_ϕ^* maps $H^2(\mathbb{D})$ onto itself. □

The following result is the first goal of this paper.

Theorem 5. *If ϕ is the function defined in Equation (2), the Toeplitz operator T_ϕ^* is universal for $H^2(\mathbb{D})$.*

Proof. We use the Theorem of Caradus (Theorem 1 above, [6], or [7, p. 214]) to establish the result.

First, Corollary 4 shows that the range of T_ϕ^* is all of $H^2(\mathbb{D})$.

For n a non-negative integer, let $w_n = -1 + i + 2n\pi i$. Notice that each of these points is in Ω. Since σ is a Riemann map of $\mathbb{D}$ onto Ω, we let $z_n = \sigma^{-1}(w_n)$. Although we do not need it here, $\{z_n\}$ is an interpolating sequence in the disk that converges tangentially to -1.

Now, writing K_{z_n} for the kernel for evaluation of H^2 functions at z_n, we see

$$\begin{aligned} T_\phi^*(K_{z_n}) &= \overline{\phi(z_n)}K_{z_n} = \overline{\left(e^{\sigma(z_n)} - e^{(-1+i)}\right)}K_{z_n} = \overline{\left(e^{(-1+i+2n\pi i)} - e^{(-1+i)}\right)}K_{z_n} \\ &= \overline{\left(e^{(-1+i)} - e^{(-1+i)}\right)}K_{z_n} = 0 \end{aligned}$$

which means for each n, the functions K_{z_n} are in the kernel of T_ϕ^*. But the functions K_{z_n} are linearly independent, so the kernel of T_ϕ^* is infinite dimensional.

By Caradus' Theorem, T_ϕ^* is a universal operator for $H^2(\mathbb{D})$. □

As has been noted, this is not the first adjoint of an analytic Toeplitz operator that has been proved to be universal. The operator T_ϕ is very different from these other adjoints of analytic Toeplitz operators that are universal because the mappings associated with these Toeplitz operators are covering maps, or close to it, and the boundary of the disk is mapped to the boundary of the set of eigenvalues of the adjoint Toeplitz operator. The mapping ϕ associated with this Toeplitz operator, however, is far from that! Indeed, the image of the unit circle, with the exception of a single point, is *inside* the (open) image of the unit disk and not on the boundary. The authors believe that it is this property that leads to the existence of a compact operator commuting with the adjoint Toeplitz operator and the possibility of the current example.

3. A Compact Weighted Composition Operator That Commutes with the Universal Operator

The goal of this section is to construct a weighted composition operator that commutes with the analytic Toeplitz operator T_ϕ. The following lemma answers the question 'When does a weighted composition operator commute with an analytic Toeplitz operator?'

Lemma 6. *For ϕ and ψ in H^∞ and J an analytic map of the unit disk into itself, the analytic Toeplitz operator T_ϕ commutes with the composition operator C_J or the weighted composition operator $W_{\psi,J}$ if and only if $\phi \circ J = \phi$.*

Proof. The cases J constant or ϕ constant are trivial, so we assume neither is constant. It is easy to see that two analytic Toeplitz operators commute with each other. Since $W_{\psi,J} = T_\psi C_J$, it is enough to check the statement for T_ϕ and C_J. We have $(T_\phi C_J h)(z) = \phi(z)h(J(z))$ and $(C_J T_\phi h)(z) = (C_J \phi h)(z) = \phi(J(z))h(J(z))$. These are equal, for all z in $\mathbb{D}$, for a non-zero h if and only if $\phi(z) = \phi(J(z))$. □

The first main result of Cowen's paper [10, p. 172] is that there is a compact weighted composition operator that commutes with T_ϕ. We recall this operator here. Let J be the analytic map of the unit disk into itself given by

$$J(z) = \sigma^{-1}(\sigma(z) + 2\pi i) \tag{3}$$

where σ is the map of the disk into the plane given by Equation (1). From this definition, an easy calculation shows that $\phi \circ J = \phi$:

$$\phi(J(z)) = e^{\sigma(\sigma^{-1}(\sigma(z)+2\pi i))} - e^{\sigma(0)} = e^{(\sigma(z)+2\pi i)} - e^{\sigma(0)} = e^{\sigma(z)} - e^{\sigma(0)} = \phi(z) \tag{4}$$

Letting $\psi(z) = (z+1)/2$, we see that ψ is continuous on the closed disk, $\psi(-1) = 0$, and $\|T_\psi\| = \|\psi\|_\infty = 1$. Finally, it is shown in [18, p. 2896] (see also [10, p. 172]) that $W_{\psi,J} = T_\psi C_J$ is a compact weighted composition operator on H^2. (Note: In [10], the multiplication operator was on the right. We have put T_ψ on the left, as is more common today, so the operator here is slightly different from that of the older paper: $C_J T_\psi = T_{\psi\circ J} C_j$, but analogous in action.)

In the next section, with a sequence of lemmas, we prove the following result that we need for the main result of this section. For the sake of keeping the important ideas together, we will state it here and use it now to prove Theorem 7.

Lemma 13. *For J as in Equation (3), the operator C_J has dense range in H^2.*

Thus, we have the following, perhaps surprising, result that is the main result of this paper:

Theorem 7. *The operator T_ϕ^* is a universal operator in the sense of Rota and T_ϕ^* commutes with $W_{\psi,J}^*$, a quasi-nilpotent, injective compact operator with dense range.*

Proof. Theorem 5 shows that T_ϕ^* is a universal operator in the sense of Rota. By Lemma 6 and Equation (4), we see that $W_{\psi,J}$ and T_ϕ commute, so $W_{\psi,J}^*$ and T_ϕ^* commute. We noted above that $W_{\psi,J}$ is compact, so $W_{\psi,J}^*$ is also compact.

The operator T_ψ is multiplication by an analytic function on the disk, so $T_\psi f = 0$ implies $f = 0$. Similarly, J is a non-constant analytic map of the disk into itself, so $C_J f = f \circ J = 0$ implies $f = 0$. This means that the weighted composition operator $W_{\psi,J} = T_\psi C_J$ is injective, and therefore $W_{\psi,J}^*$ has dense range.

In addition, the range of T_ψ is dense in H^2 because ψ is an outer function and the range of C_J is dense in H^2 by Lemma 13. This means that $W_{\psi,J}$ also has dense range which implies that $W_{\psi,J}^*$ is injective.

Finally, Lemma A of [8, p. 26] shows that, because $W_{\psi,J}$ is compact and commutes with T_ϕ, the operator $W_{\psi,J}$ is quasi-nilpotent. Therefore, $W_{\psi,J}^*$ is also quasi-nilpotent and the Theorem is proved. □

In fact, Theorem 7 is a constructive and quite explicit version of what was described as our "Main Theorem" in the introduction.

It is hoped that this example can help in the search for a solution of the invariant subspace problem on separable, infinite dimensional Hilbert spaces. In particular, this new universal operator invites an approach to the invariant subspace problem that uses properties of operators that commute with the universal operator.

4. The Proof of Lemma 13

The proof of the convexity of $J(\mathbb{D})$, defined by Equation (3), is not difficult, but it is involved. We have separated its proof so as not to interrupt the flow of the main ideas to be presented. The convexity of $J(\mathbb{D})$ implies $W_{\psi,J}$ has dense range which implies the injectivity of $W_{\psi,J}^*$.

If we let $\zeta = \sigma(z) = \dfrac{-1+i}{\sqrt{z+1}}$ then $\sigma^{-1}(\zeta) = \dfrac{-2i}{\zeta^2} - 1$ and, for z in $\mathbb{D}$,

$$J(z) = \frac{-2i}{(\sigma(z)+2\pi i)^2} - 1 = \frac{z+1}{1+2\pi(1-i)\sqrt{z+1} - 2i\pi^2(z+1)} - 1 \tag{5}$$

From Equation (5), we see that J can be extended to a continuous map of the closed disk into itself and this map satisfies $J(\overline{\mathbb{D}}) \cap \partial\mathbb{D} = \{-1\} = \{J(-1)\}$. While this fact has no relevance for our work, the set $J(\mathbb{D})$ is smaller than one might expect: $J(1) \approx -.99 + .04i$ and $|J(1) - J(-1)| \approx .04$. The boundary of the region $J(\mathbb{D})$ is shown in Figure 2 with the four points $J(-1) = -1$, $J(-i)$, $J(1)$, and $J(i)$ marked with '×'. For orienting, a portion of the unit circle near -1 is indicated by the dotted curve, and the scales on the real and imaginary axes are indicated.

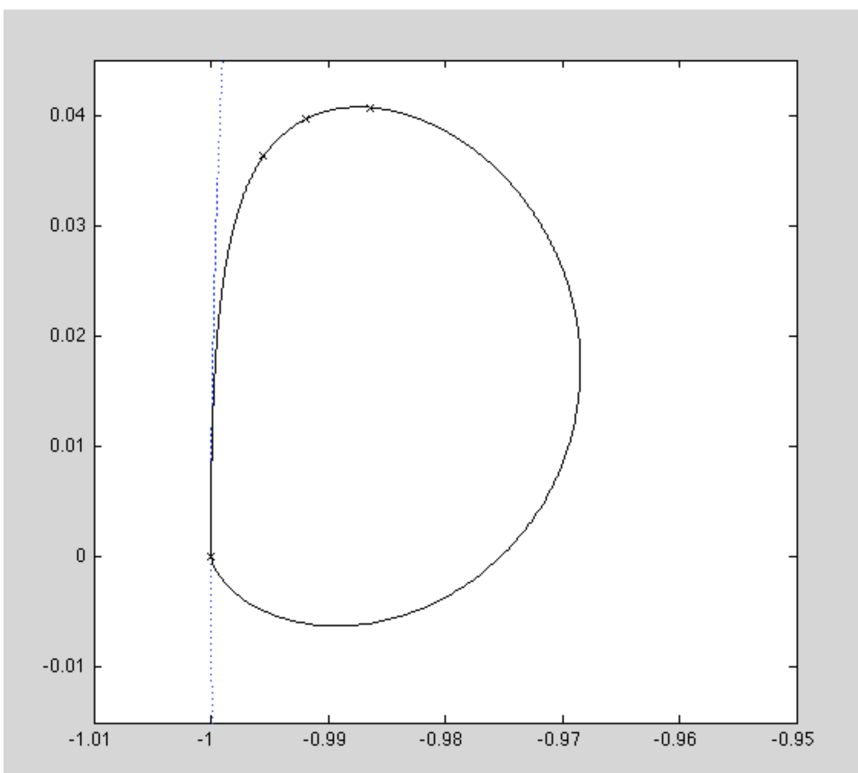

Figure 2. The set $J(\partial\mathbb{D})$ with $J(-1) = -1$, $J(-i)$, $J(1)$, and $J(i)$.

From the figure, the open set $J(\mathbb{D})$ appears to be convex. It will be helpful if we prove that, in fact, it is! There are standard inequality criteria for a univalent map of the disk onto a region to prove that the image region is convex, for example, see [22, p. 224, Prob. 5] or [24, p. 66]. Since we have the map J explicitly, this approach works, but it is rather complicated and does not give any insight. Instead, we present a proof that is more geometric and provides more insight into the convexity. We're grateful to Isabelle Chalendar for suggesting this general approach to us.

In fact, since $J(z) = \sigma^{-1}(\sigma(z) + 2\pi i)$, our strategy is to write the region $\sigma(\mathbb{D}) + 2\pi i$ as the intersection of some 'triangular regions' containing it, observe that $J(\mathbb{D})$ is the intersection of the images under σ^{-1} of these triangular regions, and, finally, show that the image under σ^{-1} of each triangular region is convex. Since the non-empty intersection of convex sets is convex, this will show that $J(\mathbb{D})$ is a convex set. We begin with some geometric lemmas.

Equation (1) gives a formula for the map σ from $\mathbb{D}$ into the plane and we observed that the image $\sigma(\mathbb{D})$ is region in the second quadrant above the branch of the hyperbola with Cartesian equation $2xy = -1$. It follows that if we raise this region by adding γi for some positive real number γ, then the resulting region will be the region in the second quadrant above the hyperbola $2x(y - \gamma) = -1$, so this is the region $\sigma(\mathbb{D}) + 2\pi i$ when we choose $\gamma = 2\pi$. For $\gamma > 0$, we will let H_γ denote the branch in the second quadrant of the hyperbola whose Cartesian equation is $2x(y - \gamma) = -1$. If $a < 0$, then the point $(a, -1/(2a) + \gamma)$ is on the hyperbola and the slope of the line tangent to the hyperbola H_γ at that point is $1/(2a^2)$. Since H_γ is in the second quadrant, every point on this branch of the hyperbola can be described in this way.

Lemma 8. *For each point, p, on the hyperbola H_γ, that is, the curve $(x, -1/(2x) + \gamma)$ for $x < 0$, there is a unique circle passing through* 0 *and tangent to the hyperbola at p.*

Proof. Notice that if $p = (a, -1/(2a) + \gamma)$, then the center of every circle that is tangent to hyperbola at p must lie on the line, ℓ_p, that passes through p and has slope $-2a^2$ because it is noted above that the tangent line has slope $1/(2a^2)$. The point 0 is in the half-plane below and to the right of this tangent line. There is a unique circle with center on ℓ_p and passing through p and 0, namely the one whose center is at the intersection of ℓ_p and the perpendicular bisector of the line segment $[0, p]$ and with radius the distance of the intersection point to 0. □

Let Z_p denote the circle described in Lemma 8. For each point p on the hyperbola, the circle Z_p either intersects both the positive imaginary axis and the negative real axis *and* at 0 (when p is relatively near 0), or intersects just one axis and 0 (when p is relatively far from 0).

Definition: For each point p on the hyperbola, let Δ_p denote the *triangular region* consisting of the (open) second quadrant in $\mathbb{C}$ intersected with the (open) exterior of the circle Z_p.

For each p on the hyperbola, the boundary of the triangular region Δ_p has three parts, unbounded intervals of the negative real axis and the positive imaginary axis, and an arc of the circle Z_p.

Lemma 9. *For $\gamma = 2\pi$, the region $\sigma(\mathbb{D}) + 2\pi i$ satisfies*

$$(\sigma(\mathbb{D}) + 2\pi i) = \bigcap_{p \in H_\gamma} \Delta_p$$

Proof. The (open) second quadrant is the union of the (open) rays starting at zero and passing through some point of the open second quadrant. If q is a point of the hyperbola H_γ, it is also clear that because the closed disk consisting of the circle Z_q and its interior is *not* in Δ_q and this disk is convex, that the line segment $[0, q]$ from 0 to q, both on the circle Z_q, is in the closed disk. That is, no part of this closed segment is in the intersection above. On the other hand, the open ray starting at q and included in the ray starting at 0 through q is in the intersection because, this entire ray is above every line tangent to the hyperbola H_γ and for each p, the circle Z_p is below the tangent at p, and therefore, for each p, the triangular region Δ_p includes this open ray. Since $(\sigma(\mathbb{D}) + 2\pi i)$ is the union of these open rays, this proves the result. $\square$

In the above, we have described the set $\sigma(\mathbb{D}) + 2\pi i$ as an intersection of the triangular regions Δ_p. Since $J(\mathbb{D})$ is $\sigma^{-1}(\sigma(\mathbb{D}) + 2\pi i)$ we see that $J(\mathbb{D})$ is the intersection of the sets $\sigma^{-1}(\Delta_p)$. It is these we want to investigate. From the calculations at the beginning of this section, we recall $\sigma^{-1}(\zeta) = \frac{-2i}{\zeta^2} - 1$. Since we are interested in convexity, we can extract the critical parts of σ^{-1} to study. First, note that convexity is translation invariant, so the translation -1 does not affect the convexity. Similarly, the rotation and dilation that are multiplication by $-2i$ do not affect convexity. We are left with understanding how the map $\zeta \mapsto \zeta^{-1}$ affects regions and how $\beta \mapsto \beta^2$ affects regions.

Lemma 10. *For p on the hyperbola H_γ, the image of the triangular region Δ_p under the map $\zeta \mapsto 1/\zeta$ is either a Euclidean triangle (if 0 is not a boundary point of Δ_p) or an unbounded triangle with one line segment and two rays as sides (if 0 is a boundary point of Δ_p). In particular, the image includes intervals ending in 0 on both the negative real and negative imaginary axes and is contained in the third quadrant.*

Proof. The image of an unbounded closed interval on the negative real axis is a closed interval including 0 on the negative real axis. The image of an unbounded closed interval on the positive imaginary axis is a closed interval including $0i = 0$ on the negative imaginary axis. In particular, since both rays in the boundary of Δ_p are unbounded, there is a vertex of the image at 0 including segments on the negative real and negative imaginary axes.

Since the map $\zeta \mapsto 1/\zeta$ is a linear fractional map, 'circles are mapped to circles', but since the the circle Z_p includes 0, the image 'circle' includes infinity and 'circles' through infinity are Euclidean lines. Thus, the arc of the circle Z_p that is included in the boundary of Δ_p is mapped to a line segment or ray (if the arc includes 0) in the plane. It is easily checked that the image of a set in the second quadrant, under this map, gets mapped to the third quadrant. $\square$

We need one final geometric result.

Lemma 11. *Suppose ℓ is a line in the complex plane that does not pass through 0 and suppose z_0 is the closest point of ℓ to 0. Then $\{z^2 : z \in \ell\}$ is a parabola having*

0 *'inside' the parabola and with axis* $\omega = \{\lambda z_0^2 : \lambda \in \mathbb{R}\}$, *that is, it is the line through* 0 *and* z_0^2.

Proof. We will give the proof for the ℓ through $z_0 = 1$ and parallel to the imaginary axis, that is, the line ℓ is parametrized by $z = 1 + it$, for t real. In this case, the set $\{z^2 : z \in \ell\}$ is parametrized by

$$z^2 = (1 + it)^2 = 1 + 2iy - t^2 = (1 - t^2) + (2t)i$$

and the Cartesian equations are $x = 1 - (y/2)^2$ which is the parabola passing through $z_0^2 = 1$ and with the real axis as the axis of the parabola and opening to the left, that is, with 0 'inside' the parabola. The general case follows from this case by dilation by a positive real number and rotation around the origin. □

We are now ready to prove the convexity we seek.

Proposition 12. *For* J *defined by Equation (3), the image* $J(\mathbb{D})$ *is a convex subset of* $\mathbb{D}$.

Proof. As noted in Lemma 9, the region above the hyperbola H_γ for $\gamma = 2\pi$ is $(\sigma(\mathbb{D}) + 2\pi i) = \bigcap_{p \in H_\gamma} \Delta_p$. Since $J(\mathbb{D}) = \sigma^{-1}(\sigma(\mathbb{D}) + 2\pi i)$ and we know that $J(\mathbb{D})$ is a non-empty open subset of $\mathbb{D}$, it is enough to prove that for each set Δ_p, $\sigma^{-1}(\Delta_p)$ is convex.

Lemma 10 shows that the image of Δ_p under the map $\zeta \mapsto 1/\zeta$ is a generalized triangle with vertex at 0 and sides along the negative real and negative imaginary axis. The discussion above makes it clear that we need to consider the image of such triangles under the map $\beta \mapsto \beta^2$.

First, under the squaring map, an interval along the negative real axis with end point at 0 is mapped to an interval along the positive real axis with end point at 0. Also, under the squaring map, an interval along the negative imaginary axis is mapped to the negative real axis. In addition, the third quadrant is mapped to the upper half plane under the squaring map.

Now the third side of the image of Δ_p is a line segment (or ray) that does not pass through 0. According to Lemma 11, this line segment (or ray) is mapped to part of a parabola. Thus, the image of Δ_p under the map $\zeta \mapsto 1/\zeta^2$ includes the point 0 in a open interval on the real line on the boundary of the image. Moreover, the real line cuts the interior of the parabola into two pieces, and the image of Δ_p under this map is one of those pieces. But *both* pieces are convex, so the image of Δ_p under this map is convex, and the image $\sigma^{-1}(\Delta_p)$ is convex for every p in H_γ.

Therefore, the intersection of these, which is $J(\mathbb{D})$, is a convex set also, as we desired to prove. □

Lemma 13. *For* J *as above, the operator* C_J *has dense range in* H^2.

Proof. We observe first that the definition of σ as a Riemann map shows J is univalent on $\mathbb{D}$. By Proposition 12, $J(\mathbb{D})$, the image of the disk under J, is a convex set. This implies that $J(\mathbb{D})$ is the complement of the closure of the unbounded component of the complement of the closure of $J(\mathbb{D})$.

It follows, from a theorem of O. J. Farrell [15, 16] (or see Sarason's paper [30, p. 521] and [29]), that the polynomials in J are weak-star dense in H^∞. In particular, because C_J acting on a polynomial in z is a polynomial in J and because H^∞ is dense in H^2, this implies C_J has dense range in H^2. □

In the previous section, we used this result to prove that the operator $W^*_{\psi,J}$ is an injective compact operator with dense range that commutes with the universal operator T^*_ϕ. This yields our Main Theorem

Main Theorem. *There exists a universal operator for separable, infinite dimensional Hilbert spaces that commutes with an injective compact operator with dense range.*

5. Consequences and Further Observations

We have seen that some universal operators commute with a compact operator and others do not. We want to try to exploit the existence of a compact operator that commutes with a particular universal operator when there is one. Our first observation is that there are many more compact operators than just the one exhibited above that commute with the universal operator T^*_ϕ. The algebra of bounded operators on the Hardy space will be denoted by $\mathcal{B}(H^2)$.

Definition: Let $\mathcal{C}$ be the set of compact operators that commute with T^*_ϕ, that is,

$$\mathcal{C} = \{G \in \mathcal{B}(H^2) : G \text{ is compact, and } \quad T^*_\phi G = G T^*_\phi\}$$

Theorem 7 above shows that $W^*_{\psi,J}$ is a non-zero operator in $\mathcal{C}$, so $\mathcal{C} \neq (0)$.

If F is a bounded operator on H^2, we will write $\{F\}'$ for the commutant of F, the set of operators that commute with F, that is, $\{F\}' = \{G \in \mathcal{B}(H^2) : GF = FG\}$. It is easy to see that for any operator F, the commutant $\{F\}'$ is a norm-closed subalgebra of $\mathcal{B}(H^2)$. We see that $\{T^*_\phi\}'$ includes $\{T^*_g : g \in H^\infty\}$ and by definition, $\mathcal{C}$ is a subset of $\{T^*_\phi\}'$. The following result gives further, surprising properties of $\mathcal{C}$.

Theorem 14. *The set $\mathcal{C}$ is a closed subalgebra of $\{T^*_\phi\}'$ that is a two-sided ideal in $\{T^*_\phi\}'$. In particular, if G is a compact operator in $\mathcal{C}$ and g and h are bounded analytic functions on the disk, then $T^*_g G$, $G T^*_h$, and $T^*_g G T^*_h$ are all in $\mathcal{C}$. Moreover, every operator in $\mathcal{C}$ is quasi-nilpotent.*

Proof. Linear combinations and products of compact operators are compact and linear combinations and products of operators that commute with T^*_ϕ also commute with T^*_ϕ, so $\mathcal{C}$ is an algebra. If G_n is a sequence in $\mathcal{C}$ such that $\lim_{n\to\infty} G_n = G$, then since each G_n is compact, G is compact and since each G_n is in the commutant of T^*_ϕ, G is also. This means that G is in $\mathcal{C}$ and $\mathcal{C}$ is norm-closed.

If F is in $\{T^*_\phi\}'$ and G is in $\mathcal{C}$, then FG and GF are compact operators in $\{T^*_\phi\}'$, so they are also in $\mathcal{C}$ and $\mathcal{C}$ is a two-sided ideal in $\{T^*_\phi\}'$. If f and g are analytic Toeplitz operators then T^*_f and T^*_g are in $\{T^*_\phi\}'$, so the assertions in the theorem about those operators follow from the more general result that $\mathcal{C}$ is an ideal.

Lemma A of [8, pg. 26] says that if R is a compact operator that commutes with an analytic Toeplitz operator, then R is quasi-nilpotent. Since T_ϕ is an analytic Toeplitz operator, if G is in $\mathcal{C}$, then $R = G^*$ commutes with T_ϕ. This means R is quasi-nilpotent so, of course, G is also quasi-nilpotent. That is, every operator in $\mathcal{C}$ is quasi-nilpotent. □

For some purposes, since we are considering A to be the restriction of T^*_ϕ to one of its invariant subspaces, it is convenient to write H^2 as the direct sum of this invariant subspace with its orthogonal complement, $H^2 = M \oplus M^\perp$. This permits every bounded operator on H^2 to be given by a block representation with respect to this splitting. Since M is invariant for T_ϕ^*, the block matrix for T_ϕ^* is upper triangular. We will denote the operators $T = T_\phi^*$ and $W = W_{\psi,J}^*$ by the block matrices:

$$T = T_\phi^* \sim \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \quad \text{and} \quad W = (T_\psi C_J)^* \sim \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \tag{6}$$

The fact that $T = T_\phi^*$ and $W = W_{\psi,J}^*$ commute gives information about the interactions between the entries of these two matrices:

$$\begin{aligned} WT \quad &\sim \quad \begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} PA & PB+QC \\ RA & RB+SC \end{pmatrix} \\ TW \quad &\sim \quad \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} AP+BR & AQ+BS \\ CR & CS \end{pmatrix} \end{aligned}$$

Equating these two computations, we see

$$PA = AP + BR \tag{7}$$
$$RA = CR \tag{8}$$

Since A is the operator of primary interest, Equation (7) is not so interesting if $P = 0$. The following Lemma says we can always avoid this situation by replacing T_ϕ^* and $W_{\psi,J}^*$ by $\widetilde{T}$ and $\widetilde{W}$ where P is not 0, but at the cost of having T and W being similar to, but not necessarily being, adjoints of an analytic Toeplitz operator and weighted composition operator, respectively. After the statement and proof of this lemma, we will continue to call them T and W, as above.

Lemma 15. *If the universal operator $T = T_\phi^*$ and the compact operator $W = W_{\psi,J}^*$ have the representations of Equation (6) with respect to the decomposition $H^2 = M \oplus M^\perp$, then there are a universal operator $\widetilde{T}$ that has M as an invariant subspace such that the restriction of $\widetilde{T}$ to M is also A and an injective compact operator $\widetilde{W}$ with dense range that commutes with $\widetilde{T}$ for which $\widetilde{P}$, the compression of $\widetilde{W}$ to M, is not zero. In other words, without loss of generality, we may assume the operator P in Equation (6) is not zero.*

Proof. If P is non-zero, then the matrices of Equation (6) satisfy the conclusion with the given description. Thus, for the rest of this proof, we may assume $P = 0$ in Equation (6). Since we know $\ker(W) = (0)$, we know that because $P = 0$ then $R \neq 0$ and, indeed, $\ker(R) = (0)$.

We will show there is an invertible operator on H^2 which gives a similarity between T and $\widetilde{T}$ and W and $\widetilde{W}$ so that the resulting $\widetilde{P}$ is not 0.

Using the splitting $H^2 = M \oplus M^\perp$, we want to choose $X : M^\perp \mapsto M$ such that the similarity induced by $\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$ gives the conclusion.

$$\widetilde{T} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} A & B + XC - AX \\ 0 & C \end{pmatrix} \tag{9}$$

$$\widetilde{W} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & Q \\ R & S \end{pmatrix}\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} = \begin{pmatrix} XR & -XRX + Q + XS \\ R & -RX + S \end{pmatrix} \tag{10}$$

Since $R \neq 0$, we can choose any X that is not 0 on the range of R. □

The following are immediate corollaries of the existence of the injective compact operator W with dense range commuting with the universal operator T, Lemma 15, and the above calculations. Recall that a closed subspace L on a Hilbert space $\mathcal{H}$ is said to be a *nontrivial hyperinvariant subspace* for the bounded operator F if $L \neq (0)$, $L \neq \mathcal{H}$, and L is invariant for every operator that commutes with F. Lomonosov's Theorem [21] can be stated as *Non-scalar operators that commute with a nontrivial compact operator have hyperinvariant subspaces.*

Theorem 16. *Let the universal operator T and the commuting injective compact operator W with dense range have the representations of Equation (6) with $P \neq 0$. Then the following are true:*

[I] *Either $R \neq 0$ or A has a nontrivial hyperinvariant subspace.*
[II] *Either ker$(R) = (0)$ or A has a nontrivial invariant subspace.*
[III] *Either $B \neq 0$ or A has a nontrivial hyperinvariant subspace.*

Proof.

[I] If $R = 0$, then Equation (7) says $PA = AP + B \cdot 0 = AP$. Since $R = 0$ implies $P \neq 0$, Lomonosov's Theorem implies A has a nontrivial hyperinvariant subspace.
[II] If $R \neq 0$ but $\ker(R) \neq (0)$, then $\ker(R)$ is a nontrivial subspace of M and Equation (8) shows that if x is in $\ker(R)$, then $R(Ax) = C(Rx) = 0$ which implies $\ker(R)$ is an invariant subspace for A.
[III] If $B = 0$, then Equation (7) says $PA = AP + 0 \cdot R = AP$. Lemma 15 shows that without loss of generality, $P \neq 0$ and Lomonosov's Theorem implies A has a nontrivial hyperinvariant subspace.

□

If v is an eigenvector for T_ϕ^* with eigenvalue λ, then we can write $v = (x, y)$ where x is the projection of v onto M and y is the projection of v onto $M^\perp$. Then representation above for T_ϕ^* makes it easy to see, for example, that y is an eigenvector for C. The following lemma is a generalization of this statement.

Proposition 17. *Suppose L is an invariant subspace for T_ϕ^* and the block matrix in Equation (6) represents T_ϕ^* based on the splitting $H^2 = M \oplus M^\perp$. Then the projection of the invariant subspace L onto $M^\perp$ is an invariant linear manifold for C, the compression of T_ϕ^* to $M^\perp$.*

Proof. Suppose v is a vector in L and x is the projection of v onto M and y is the projection of v onto $M^\perp$. Then we see that

$$T_\phi^* v = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} Ax + By \\ Cy \end{pmatrix}$$

which shows that the projection of $T_\phi^* v$ onto $M^\perp$ is Cy. Since $T_\phi^* v$ is a vector in L because L is invariant for T_ϕ^*, this says that for every y in the projection of L onto $M^\perp$, the vector Cy is also in the projection of L onto $M^\perp$. □

In fact, as a corollary of this result we have:

Corollary 18. *Suppose L is an invariant subspace for T_ϕ^* and the block matrix in Equation 6 represents T_ϕ^* based on the splitting $H^2 = M \oplus M^\perp$. Then the projection of $L^\perp$ onto M is an invariant linear manifold for A^*, the adjoint of the restriction of T_ϕ^* to M.*

Proof. Since L is invariant for the operator T_ϕ^*, the subspace $L^\perp$ is invariant for the operator $T_\phi = (T_\phi^*)^*$. Now A^* is the compression of $(T_\phi^*)^*$ to $M = (M^\perp)^\perp$, so the conclusion is just an application of the proof of the lemma to T_ϕ and $L^\perp$. □

Observe that any of the linear manifolds provided by Corollary 18 are proper and invariant but, in principle, they are not necessarily non-dense. We believe that the examples described here are interesting and we hope that they may be useful in the study of the *Invariant Subspace Problem* for Hilbert spaces.

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